THE CONE OF BETTI DIAGRAMS OVER A HYPERSURFACE RING OF LOW EMBEDDING DIMENSION

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ABSTRACT. We give a complete description of the cone of Betti diagrams over a standard graded hypersurface ring of the form $\Bbbk[x,y]/\langle q\rangle$, where q is a homogeneous quadric. We also provide a finite algorithm for decomposing Betti diagrams, including diagrams of infinite projective dimension, into pure diagrams. Boij–Söderberg theory completely describes the cone of Betti diagrams over a standard graded polynomial ring; our result provides the first example of another graded ring for which the cone of Betti diagrams is entirely understood.

1. INTRODUCTION

In an important shift in perspective to the study of graded free resolutions, Boij and Söderberg suggested that their numerics are easier to understand "up to scalar multiplication" [BS08]. More specifically, for minimal free resolutions of graded modules over the standard graded polynomial ring, they formulated precise conjectures about the possible Betti diagrams of such modules, including a description of the extremal rays of the cone of all such Betti diagrams. The subsequent proof of their conjectures [EFW11, ES09, BS08b, ES10] provided a breakthrough in our understanding of the structure of graded free resolutions, including a proof of the Herzog–Huneke–Srinivasan Multiplicity Conjectures [HS98, Conjectures 1 and 2].

In this paper we investigate Boij–Söderberg theory for graded hypersurface rings, where the existence of resolutions of infinite projective dimension complicates the picture. Our main result is a complete description of the cone of Betti diagrams over a standard graded hypersurface ring of the form $\Bbbk[x,y]/\langle q\rangle$, where q is a homogeneous quadric. As in the case of a standard graded polynomial ring, there is a partial order on the extremal rays of the cone which gives it the structure of a simplicial fan. We obtain a similar result for standard graded rings of the form $\Bbbk[x]/\langle x^n\rangle$ for any $n \geq 2$. Although there has been recent work on extending Boij–Söderberg theoretic results to rings other than the polynomial ring [BF11, Flø10, BBCI$^+$10, BEKS11], the main result of this paper provides the first example of another graded ring for which the cone of Betti diagrams is entirely understood.

For a standard graded Noetherian commutative ring R and a finitely generated graded R-module M with minimal graded free resolution $F_\bullet \colon F_0 \leftarrow F_1 \leftarrow \cdots$, let $F_i = \bigoplus_{j\in\mathbb{Z}} R(-j)^{\beta^R_{ij}(M)}$. Set $\mathbb{V}$ to be the space of column finite matrices with entries in $\mathbb{Q}$, and define the $\mathsf{Betti\ diagram}$ of M, denoted $\beta^R(M)$, to be the matrix whose entries are given by

$$\left(\beta^R(M)\right)_{i,j} := \beta^R_{i,j}(M) \in \mathbb{V}.$$

We adopt the standard Betti diagram convention when displaying matrices in $\mathbb{V}$, writing

$$\beta^R(M) = \begin{pmatrix} \vdots & \vdots & \vdots & \\ {}^*\beta^R_{0,0}(M) & \beta^R_{1,1}(M) & \beta^R_{2,2}(M) & \cdots \\ \beta^R_{0,1}(M) & \beta^R_{1,2}(M) & \beta^R_{2,3}(M) & \cdots \\ \vdots & \vdots & \vdots & \end{pmatrix},$$

where the symbol * identifies the $(0,0)$-entry (this symbol may be omitted when the indexing is clear from context).

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We define the cone of Betti diagrams over R to be

$$\mathrm{B}_{\mathbb{Q}}(R) := \left\{ \sum_M a_M \beta^R(M) \,\middle|\, a_M \in \mathbb{Q}_{\geq 0} \text{ and almost all } a_M \text{ are zero} \right\} \subseteq \mathbb{V},$$

so that it is the positive hull of $\beta^R(M)$ for all finitely generated graded R-modules M.

As in the case of a polynomial ring, our description of the cone of Betti diagrams for the hypersurfaces above depends on the notion of a pure resolution; this is a resolution of the form

$$R(-d_0)^{\beta_{0,d_0}} \leftarrow R(-d_1)^{\beta_{1,d_1}} \leftarrow R(-d_2)^{\beta_{2,d_2}} \leftarrow \cdots$$

for some integers $d_0 < d_1 < d_2 < \cdots$ with $d_i \in \mathbb{Z} \cup \infty$. (By convention $R(-\infty) = 0$ and $\infty < \infty$.) We refer to $(d_0, d_1, d_2, \ldots)$ as the degree sequence of the pure resolution.

The simplest hypersurface ring R is one of embedding dimension 1. The extremal ray description of this cone, provided in Proposition 1.1, follows from the structure theorem of finitely generated modules over a principal ideal domain. We give an equivalent description of this cone in terms of facets in Theorem 3.4.

Proposition 1.1. *Let $R = \Bbbk[x]/\langle x^n \rangle$. The extremal rays of $\mathrm{B}_{\mathbb{Q}}(R)$ are the rays in $\mathbb{V}$ spanned by:*

(i) the Betti diagrams of those modules of finite projective dimension having a pure resolution of the form $(d_0, \infty, \infty, \ldots)$;

(ii) the Betti diagrams of those modules of infinite projective dimension having a pure resolution of type $(d_0, d_1, d_0 + n, d_1 + n, \ldots)$.

Our main result is a complete description of the cone $\mathrm{B}_{\mathbb{Q}}(R)$ when R is a quadric hypersurface ring of embedding dimension 2. We state here its description in terms of extremal rays; see Theorem 2.4 for a description in terms of facets.

Theorem 1.2. *Let q be any quadric in $\Bbbk[x, y]$, and let $R = \Bbbk[x, y]/\langle q \rangle$. The extremal rays of $\mathrm{B}_{\mathbb{Q}}(R)$ are the rays in $\mathbb{V}$ spanned by:*

(i) the Betti diagrams of those Cohen–Macaulay modules of finite projective dimension having a pure resolution of the form $(d_0, d_1, \infty, \ldots)$;

(ii) the Betti diagrams of those finite length modules of infinite projective dimension having a pure resolution of type $(d_0, d_1, d_1 + 1, d_1 + 2, \ldots)$.

As in the main results of Boij–Söderberg theory for the standard graded polynomial ring [ES09], for both types of hypersurfaces R above, our results provide a simplicial fan structure on $\mathrm{B}_{\mathbb{Q}}(R)$ (for the definition of simplicial fan and other notions from convex geometry, see Appendix A).

Theorem 1.3. *Let R be a standard graded hypersurface ring of the form $\Bbbk[x]/\langle x^n \rangle$ for any $n \geq 2$ or $\Bbbk[x, y]/\langle q \rangle$, where q is any homogeneous quadric. Then the cone of Betti diagrams $\mathrm{B}_{\mathbb{Q}}(R)$ has the structure of a simplicial fan induced by a partial order on its extremal rays.*

From the simplicial fan structure, we obtain decomposition algorithms for R-Betti diagrams as in [ES09, BS08b], as well as R-analogues of the Multiplicity Conjectures (see §4).

New phenomena arise in the hypersurface case that are not seen in the case of a standard graded polynomial ring. To begin with, some of the functionals used to provide a halfspace description of $\mathrm{B}_{\mathbb{Q}}(R)$ have no analogue in the polynomial ring case. One set of these functionals directly measures the nonminimality of the standard resolution. This resolution, introduced in [Sha69, §3] (see also [Eis80, §7]), builds a free R-resolution from a minimal free S-resolution. The resulting functionals thus directly reflect the passage from the polynomial to hypersurface case.

Another interesting difference comes from the simplicial structure on $\mathrm{B}_{\mathbb{Q}}(R)$. Unlike the polynomial ring, we cannot simply use the termwise partial order on R-degree sequences. Instead, we introduce partial orders that take into account the infinite resolutions that occur over a hypersurface ring, see Definitions 2.1 and 3.1.

Finally, we observe that for hypersurface rings, it is no longer the case that every Cohen–Macaulay module with a pure resolution lies on an extremal ray. This already happens in the context of Theorem 1.2. For instance, let M be the maximal Cohen–Macaulay module $\langle x\rangle \subseteq R := \Bbbk[x,y]/\langle x^2\rangle$. The Betti diagram of M is not extremal, as it decomposes as

$$\beta^R(M) \;=\; \begin{pmatrix} 1 & 1 & 1 & 1 & \cdots \\ - & - & - & - & \cdots \end{pmatrix} \;=\; \frac{1}{2}\begin{pmatrix} 1 & 2 & 2 & 2 & \cdots \\ - & - & - & - & \cdots \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 1 & - & - & - & \cdots \\ - & - & - & - & \cdots \end{pmatrix}.$$

Consideration of more general hypersurfaces complicates this situation even further and suggests some of the challenges in expanding this theory to hypersurfaces of higher degree.

Proposition 1.4. *Let $R = \Bbbk[x_1,\dots,x_r]/\langle f\rangle$ be any hypersurface ring with $r > 1$ and $\deg(f) > 2$. Then $\beta^R(\Bbbk)$ is an extremal ray in $\mathrm{B}_{\mathbb{Q}}(R)$ which is not pure.*

Proof. The Tate resolution of $\Bbbk$, introduced in [Tat57], is the minimal free resolution of $\Bbbk$. Using this resolution, since $\deg(f) > 2$ and $r > 1$, one checks that the second syzygy module $\Omega^2(\Bbbk)$ has minimal generators in degrees 2 and $\deg(f) - 1$, and thus $\beta^R(\Bbbk)$ is not pure.

We next claim that if M is any module generated in degree 0, then $\beta^R_{1,1}(M) \le r \cdot \beta^R_{0,0}(M)$ with equality if and only if M is a direct sum of copies of $\Bbbk$. To see this, we first set $a := \beta_{0,0}(M)$. The linear first syzygies of M will be linearly independent in the $\Bbbk$-vector space R_1^a; since $\dim R_1 = r$, this space is $r\cdot a$-dimensional, implying the inequality. Now, if $M \cong \Bbbk^a$, then equality holds by the Tate resolution. Conversely, if $\beta^R_{1,1}(M) = r\cdot a$ then each generator of M is annihilated by $(x_1,\dots,x_r)$, and so $M \cong \Bbbk^a$.

Finally, to see that $\beta^R(\Bbbk)$ is extremal, suppose that $\beta^R(\Bbbk) = \sum a_i\beta^R(M_i)$ for R-modules M_i and $a_i \in \mathbb{Q}_{\ge 0}$. This implies that $\sum a_i\beta^R_{0,0}(M_i) = \beta^R_{0,0}(\Bbbk) = 1$. Using this, and the claim above, we have

$$r = \beta^R_{1,1}(\Bbbk) = \sum a_i\beta^R_{1,1}(M_i) \le \sum ra_i\beta^R_{0,0}(M_i) = r$$

and so $\sum a_i\beta^R_{1,1}(M_i) = \sum ra_i\beta^R_{0,0}(M_i)$. Since $\beta^R_{1,1}(M_i) \le r\beta^R_{0,0}(M_i)$ for all i, we must have equality, which by the claim implies that each M_i is a direct sum of copies of $\Bbbk$. $\square$

This paper is outlined as follows. In §2, we consider the case of a quadric hypersurface ring with embedding dimension 2. §3 is dedicated to the case of embedding dimension 1. §4 addresses Theorem 1.3 and the R-analogues of the Multiplicity Conjectures. Finally, we include Appendix A on convex geometry.

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2. Resolutions over hypersurface rings of embedding dimension 2 and degree 2

Set $S := \Bbbk[x,y]$ and $R := S/\langle q\rangle$ for a quadric q in S. In this section, we give a full description of the cone of Betti diagrams of R-modules, including a proof of Theorem 1.2.

Definition 2.1. We say that

$$d = (d_0, d_1, d_2, \dots) \in \prod_{i\in\mathbb{N}}(\mathbb{Z}\cup\{\infty\})$$

is an R-degree sequence if it has the form

(i) $d = (d_0, \infty, \infty, \infty, \dots)$,
(ii) $d = (d_0, d_1, \infty, \infty, \dots)$ with $d_0 < d_1$, or
(iii) $d = (d_0, d_1, d_1 + 1, d_1 + 2, \dots)$ with $d_0 < d_1$.

We define a partial order $\leq$ on R-degree sequences as follows. We do a termwise comparison on the first two entries; in the case of a tie, we then do a termwise comparison on the remaining entries. In other words, for two R-degree sequences d, d' we say that $d \leq d'$ if either

- $d_0 \leq d'_0$ and $d_1 \leq d'_1$, with one of these inequalities being strict, or
- $d_0 = d'_0$, $d_1 = d'_1$, and $d_n \leq d'_n$ for all $n \geq 2$.

Definition 2.1 leads to a decomposition algorithm (see §4) and fits into the framework of [BEKS10].

Recall that the $\mathbb{Q}$-vector space $\mathbb{V}$ is the set of column-finite matrices with columns indexed by $i \in \mathbb{N}$ and rows indexed by $j \in \mathbb{Z}$. For each R-degree sequence d, we define a matrix $\pi_d \in \mathbb{V}$ as follows. Set $(\pi_d)_{0,j} = 1$ for $j = d_0$ and 0 otherwise.

(i) If $d = (d_0, \infty, \ldots)$, set $(\pi_d)_{i,j} = 0$ for all $i \geq 1$ and all j.
(ii) If $d = (d_0, d_1, \infty, \ldots)$, set $\pi_{1,j} = 1$ when $j = d_1$ and 0 otherwise.
(iii) If $d = (d_0, d_1, d_1 + 1, d_1 + 2, \ldots)$, set $\pi_{i,j} = 2$ when $i \geq 1$ and $j = d_i$, and 0 otherwise.

Example 2.2. Three degree sequences and their corresponding Betti diagrams appear below.

$$\pi_{(0,\infty,\ldots)} = \begin{pmatrix} \vdots & \vdots & \vdots & \\ - & - & - & \cdots \\ {}^*1 & - & - & \cdots \\ - & - & - & \cdots \\ - & - & - & \cdots \\ \vdots & \vdots & \vdots & \end{pmatrix} \quad \pi_{(1,2,\infty,\ldots)} = \begin{pmatrix} \vdots & \vdots & \vdots & \\ - & - & - & \cdots \\ {}^*- & - & - & \cdots \\ 1 & 1 & - & \cdots \\ - & - & - & \cdots \\ \vdots & \vdots & \vdots & \end{pmatrix} \quad \pi_{(0,3,4,5,\ldots)} = \begin{pmatrix} \vdots & \vdots & \vdots & \\ - & - & - & \cdots \\ {}^*1 & - & - & \cdots \\ - & - & - & \cdots \\ - & 2 & 2 & \cdots \\ \vdots & \vdots & \vdots & \end{pmatrix}$$

We define functionals on $v \in \mathbb{V}$ as follows:

$$\epsilon_{i,j}(v) := v_{i,j}, \quad \alpha_k(v) := \epsilon_{1,k}(v) - \epsilon_{2,k+1}(v), \quad \text{and} \quad \gamma_k(v) := \sum_{j \leq k} \left(2\epsilon_{0,j}(v) - 2\epsilon_{1,j+1}(v) + \epsilon_{2,j+2}(v)\right).$$

Observe that the functional γ_k is well-defined for any $v \in \mathbb{V}$ because v is column-finite.

Example 2.3. The functional γ_2 applied to a Betti diagram $\beta^R(M)$ is given by taking the dot product of $\beta^R(M)$ with the following diagram:

$$\begin{pmatrix} \vdots & \vdots & \vdots & \vdots & \\ {}^*2 & -2 & 1 & 0 & \cdots \\ 2 & -2 & 1 & 0 & \cdots \\ 2 & -2 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{pmatrix}.$$

Theorem 2.4. *The following cones in $\mathbb{V}$ are equal:*

(i) The cone $\mathrm{B}_{\mathbb{Q}}(R)$ spanned by the Betti diagrams of all finitely generated R-modules.
(ii) The cone D spanned by π_d for all R-degree sequences d.
(iii) The cone F defined to be the intersection of the halfspaces
  (a) $\{\epsilon_{i,j} \geq 0\}$ for all $i \geq 0$ and $j = 0$ or 2;
  (b) $\{\alpha_k \geq 0\}$ for all $k \in \mathbb{Z}$;
  (c) $\{\gamma_k \geq 0\}$ for all $k \in \mathbb{Z}$;
  (d) $\{\pm(\epsilon_{i,j} - \epsilon_{i+1,j+1}) \geq 0\}$ for $i \geq 2, j \in \mathbb{Z}$.

To prove Theorem 2.4, we show the inclusions $D \subseteq \mathrm{B}_{\mathbb{Q}}(R) \subseteq F \subseteq D$ which are contained in Lemmas 2.5, 2.6, and 2.8, respectively. The proof of Lemma 2.5 is straightforward, and the proof of Lemma 2.8 largely follows Boij and Söderberg's techniques involving convex polyhedral geometry. By contrast, the proof of Lemma 2.6 requires new ideas. In particular, we use a construction due

to [Sha69] that constructs a (not necessarily minimal) R-free resolution from an S-free resolution; see also [Eis80, §7]. We briefly recall this construction now.

Let $G_\bullet$ be a graded free S-resolution of an R-module M (recall that $S = \Bbbk[x,y]$) Since multiplication by q is nullhomotopic on $G_\bullet$, there are homotopy maps s_1, s_2:

$$\begin{tikzcd}
0 & G_0(-2) \arrow[l] \arrow[d, "q"'] \arrow[dr, "s_1"] & G_1(-2) \arrow[l] \arrow[d, "q"'] \arrow[dr, "s_2"] & G_2(-2) \arrow[l] \arrow[d, "q"'] & 0 \arrow[l] \\
0 & G_0 \arrow[l] & G_1 \arrow[l] & G_2 \arrow[l] & 0. \arrow[l]
\end{tikzcd}$$

Now set $\overline{G}_i := G_i \otimes R$, $\overline{\partial}_i = \partial_i \otimes R$, and $\overline{s}_i := s_i \otimes R$ for $i = 1, 2$. The resulting complex

$$0 \longleftarrow \overline{G}_0 \xleftarrow{\overline{\partial}_1} \overline{G}_1 \xleftarrow{\left(\overline{\partial}_2\,,\,\overline{s}_1\right)} \begin{matrix}\overline{G}_2\\ \oplus \\ \overline{G}_0(-2)\end{matrix} \xleftarrow{\left(\begin{smallmatrix}\overline{s}_2\\ \overline{\partial}_1\end{smallmatrix}\right)} \overline{G}_1(-2) \xleftarrow{\left(\overline{\partial}_2\,,\,\overline{s}_1\right)} \begin{matrix}\overline{G}_2(-2)\\ \oplus \\ \overline{G}_0(-4)\end{matrix} \longleftarrow \cdots$$

is an R-free resolution of M. Note that there are additional maps $G_i \to G_{i+2d-1}$ in the construction given in [Sha69, §3]. These maps are 0 in our context because $G_i = 0$ when $i \geq 3$.

Lemma 2.5. *There is an inclusion $D \subseteq \mathrm{B}_{\mathbb{Q}}(R)$.*

Proof. It suffices to show that, for each R-degree sequence d, there exists an R-module M with $\beta^R(M) = \pi_d$. If $d = (d_0, \infty, \dots)$, we simply choose $M = R(-d_0)$. For the other cases, fix ℓ, a linear form not a scalar multiple of x, that is a nonzero divisor on R (i.e., ℓ does not divide q). Such an ℓ exists in any characteristic. If $d = (d_0, d_1, \infty, \dots)$, we set $M = R(-d_0)/\langle \ell^{d_1-d_0}\rangle$.

Finally, if $d = (d_0, d_1, d_1+1, d_1+2, \dots)$, we set M to be $R(-d_0)/\langle \ell^{d_1-d_0}, x\ell^{d_1-d_0-1}\rangle$. To see that M has the desired Betti diagram, we first consider the minimal S-free resolution. By hypothesis $q, \ell^{d_1-d_0}, x\ell^{d_1-d_0-1}$ are a minimal set of generators in S. Applying the Hilbert-Burch theorem, see e.g. [Eis95, 20.15], the S-free resolution of M has the form:

$$0 \longleftarrow S(-d_0) \xleftarrow{\partial_1} \begin{matrix}S(-d_0-2)\\ \oplus \\ S(-d_1)^2\end{matrix} \longleftarrow S(-d_1-1)^2 \longleftarrow 0.$$

where $\partial_1 = \begin{bmatrix} q & \ell^{d_1-d_0} & x\ell^{d_1-d_0-1}\end{bmatrix}$. We fix homotopies s_1, s_2 for multiplication by q on this resolution:

$$\begin{tikzcd}
0 & S(-d_0-2) \arrow[l] \arrow[d, "q"'] \arrow[dr, "s_1"] & \begin{matrix}S(-d_0-4)\\ \oplus \\ S(-d_1-2)^2\end{matrix} \arrow[l] \arrow[d, "q"'] \arrow[dr, "s_2"] & S(-d_1-3)^2 \arrow[l] \arrow[d, "q"'] & 0 \arrow[l] \\
0 & S(-d_0) \arrow[l] & \begin{matrix}S(-d_0-2)\\ \oplus \\ S(-d_1)^2\end{matrix} \arrow[l, "\partial_1"'] & S(-d_1-1)^2 \arrow[l] & 0. \arrow[l]
\end{tikzcd}$$

Since ℓ does not divide q we see that the component of s_1 that maps $S(-d_0-2)$ to $S(-d_0-2)$ must be 1. By degree considerations, the maps s_1 and s_2 cannot contain any other unit entries.

The standard resolution of M is now given by

$$0 \longleftarrow R(-d_0) \longleftarrow \begin{matrix}R(-d_0-2)\\ \oplus \\ R(-d_1)^2\end{matrix} \longleftarrow \begin{matrix}R(-d_1-1)^2\\ \oplus \\ R(-d_0-2)\end{matrix} \longleftarrow \begin{matrix}R(-d_0-4)\\ \oplus \\ R(-d_1-2)^2\end{matrix} \longleftarrow \cdots.$$

The maps $R(-d_0-2n) \leftarrow R(-d_0-2n)$ are the only nonminimal part of this resolution. It follows that M has a minimal free R-resolution of the form

$$0 \longleftarrow R(-d_0) \longleftarrow R(-d_1)^2 \longleftarrow R(-d_1-1)^2 \longleftarrow R(-d_1-2)^2 \longleftarrow \cdots,$$

which yields the desired Betti diagram. □

Lemma 2.6. *There is an inclusion* $\mathrm{B}_{\mathbb{Q}}(R) \subseteq F$*.*

Proof. Fix a finitely generated graded R-module M. We must show that the inequalities defining F are nonnegative on $\beta^R(M)$. Certainly $\epsilon_{i,j}(\beta^R(M)) = \beta^R_{i,j}(M) \geq 0$ for all i and j, completing case (iiia). For case (iiid), the minimal resolution of M is given by a matrix factorization after at most two steps by [Eis80, Theorem 4.1]. By [Eis95, Lemma 20.11], $\Omega_2^R(M)$ has depth 2 and is thus maximal Cohen–Macaulay. After extending the base field to its algebraic closure (which does not affect Betti diagrams), the homogeneous quadric q is, without loss of generality, either x^2 or xy. The matrix factorizations of these quadrics over an algebraically closed field are classified (see [Yos90, Example 6.5 and p. 76]). Thus the resolution of M after at most 2 steps is given by one of the matrix factorizations above; one easily checks for these that (iiid) hold.

For case (iiib), we show that $\alpha_k\left(\beta^R(M)\right) \geq 0$ by showing that it measures the rank of a map. Fix a minimal free S-resolution $G_\bullet$ of M as above, and let s_1 and s_2 denote the homotopies occurring in the standard resolution of M over R. Let $\sigma_{i,j}$ be the composition of the maps

$$\sigma_{i,j} \colon S(-j)^{\beta^S_{i-1,j-2}(M)} \hookrightarrow G_{i-1}(-2) \xrightarrow{s_i} G_i \twoheadrightarrow S(-j)^{\beta^S_{i,j}(M)}.$$

With the chosen basis, the entries of $\sigma_{i,j}$ have degree 0, so $\sigma_{i,j}$ is a matrix of elements of $\Bbbk$. We claim that

$$\alpha_k(\beta^R(M)) := \beta^R_{1,k}(M) - \beta^R_{2,k+1}(M) = \operatorname{rank}\sigma_{2,k} \geq 0.$$

It follows from this construction that

$$\beta^R_{1,k}(M) = \beta^S_{1,k}(M) - \operatorname{rank}\sigma_{1,k}$$
$$\text{and}\quad \beta^R_{3,k+2}(M) = \beta^S_{1,k}(M) - \operatorname{rank}\sigma_{1,k} - \operatorname{rank}\sigma_{2,k}.$$

Thus $\beta^R_{1,k}(M) - \beta^R_{3,k+2}(M) = \operatorname{rank}\sigma_{2,k}$. As noted above in the proof of (iiid), $\beta^R_{2,k+1}(M) = \beta^R_{3,k+2}(M)$, which yields the claim.

Finally, for case (iiic), or γ_k, let $\phi_1 \colon F_1 \to F_0$ be a minimal presentation of M over R and set

$$F_0' := \bigoplus_{j\leq k} R(-j)^{\beta^R_{0,j}(M)} \qquad \text{and} \qquad F_1' := \bigoplus_{j\leq k+1} R(-j)^{\beta^R_{1,j}(M)}.$$

There are natural split inclusions $F_0' \subseteq F_0$ and $F_1' \subseteq F_1$. In particular, ϕ_1 induces a map $\phi_1' \colon F_1' \to F_0'$. We set $N := \operatorname{coker}(\phi_1')$, and note that ϕ_1' is a minimal presentation of N'. As such, $\beta^R_{0,j}(M) = \beta^R_{0,j}(N)$ for all $j \leq k$, and $\beta^R_{1,j}(M) = \beta^R_{1,j}(N)$ for all $j \leq k+1$. In addition, we claim that $\beta^R_{2,j}(M) = \beta^R_{2,j}(N)$ for all $j \leq k+2$. To see this, consider the diagram

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & \Omega_2^R(N) \arrow[r] \arrow[d] & F_1' \arrow[r] \arrow[d] & F_0' \arrow[r] \arrow[d] & N \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \Omega_2^R(M) \arrow[r] & F_1 \arrow[r] & F_0 \arrow[r] & M \arrow[r] & 0,
\end{tikzcd}$$

where we view $\Omega_2^R(N)$, $\Omega_2^R(M)$ as submodules of F_1', F_1 respectively. By the Snake Lemma, $\Omega_2^R(N)$ is a submodule of $\Omega_2^R(M)$. For a fixed basis of F_1, any element of $\Omega_2^R(M)$ may be written as a linear

combination of the basis elements with coefficients in $R_{\geq 1}$. Thus for an element $x \in \Omega_2^R(M)$ of degree j with $j \leq k+2$, we see that the basis elements whose corresponding coefficients are nonzero in a decomposition of x have degree at most $j-1 \leq k+1$. In particular, these basis elements are in F_1', and hence $\Omega_2^R(N)_j = \Omega_2^R(M)_j$ for all $j \leq k+2$, which implies the claim.

By the definition of γ_k, we have now shown that $\gamma_k\left(\beta^R(M)\right) = \gamma_k\left(\beta^R(N)\right)$. It thus suffices to show that $\gamma_k\left(\beta^R(N)\right) \geq 0$. We achieve this by showing that $\gamma_k\left(\beta^R(N)\right) = h_{k+2}(N)$, where $h_{k+2}(N)$ denotes the Hilbert function of N in degree $k+2$.

The Hilbert function of N can be computed entirely in terms of $\beta^R(N)$:

$$\begin{aligned} h_{k+2}(N) &= \sum_{j\in\mathbb{Z}}\sum_{i=0}^{\infty}(-1)^i\beta^R_{i,j}(N)h_{k+2}(R(-j)) \\ &= \sum_{j\in\mathbb{Z}}\sum_{i=0}^{\infty}(-1)^i\beta^R_{i,j}(N)h_{k+2-j}(R) \\ &= \sum_{\ell\in\mathbb{Z}}\sum_{i=0}^{\infty}(-1)^i\beta^R_{i,i+\ell}(N)h_{k+2-i-\ell}(R). \end{aligned}$$

Since $\beta^R_{0,j}(N) = 0$ for $j > k$, $\beta^R_{1,j}(N) = 0$ for $j > k+1$, and $h_i(R) = 2$ for all $i > 0$, we have that

$$\begin{aligned} h_{k+2}(N) &= \sum_{\ell\leq k}\sum_{i=0}^{\infty}(-1)^i\beta^R_{i,i+\ell}(N)h_{k+2-i-\ell}(R) \\ &= \sum_{\ell\leq k}\left(\sum_{i=0}^{k+1-\ell}(-1)^i\beta^R_{i,i+\ell}(N)\cdot 2\right) + (-1)^{k+2-\ell}\beta^R_{k+2-\ell,k+2}(N)\cdot 1. \end{aligned}$$

By applying (iiid) twice, we see that $\beta^R_{i,j}(N) = \beta^R_{i+2,j+2}(N)$ for all $i \geq 2$. Using this to cancel, we obtain

$$h_{k+2}(N) = \sum_{\ell\leq k}\left(2\beta^R_{0,\ell}(N) - 2\beta^R_{1,\ell+1}(N) + \beta^R_{2,\ell+2}(N)\right) = \gamma_k(\beta^R(N)). \qquad \square$$

For the final inclusion in the proof of Theorem 2.4, we compare the cone D (which is defined in terms of extremal rays) and the cone F (which is defined in terms of halfspaces). As we see in Lemma A.1, it is easier to move between these two descriptions in the case of a simplicial fan, so we first construct a simplicial fan Σ whose support is contained in D.

Lemma 2.7. *For every finite chain C of R-degree sequences, the cone* $\mathrm{pos}(C) := \mathbb{Q}_{\geq 0}\{\pi_d \mid d \in C\}$ *is simplicial. The collection of these simplicial cones forms a simplicial fan.*

Proof. The diagrams π_d from any finite chain C are linearly independent. This follows from the fact that for any degree sequence d, π_d has a nonzero entry in a position such that, for every degree sequence d' in the chain C with $d < d'$, $\pi_{d'}$ has a zero in the corresponding position.

For the second statement, we need to show that these cones meet along faces. Using the observation above, the proof of [BS08, 2.9] applies directly to our situation. $\square$

Lemma 2.8. *There is an inclusion $F \subseteq D$.*

Proof. Let Σ be the simplicial fan constructed in Lemma 2.7, and let $\mathrm{supp}(\Sigma)$ denote its support, as defined in Appendix A. By construction, $\mathrm{supp}(\Sigma) \subseteq D$, so it suffices to prove that $F \subseteq \mathrm{supp}(\Sigma)$.[1] Now, we have a simplicial fan Σ defined in terms of extremal rays, and we seek to determine its

[1] A priori, $\mathrm{supp}(\Sigma)$ is a (not necessarily convex) subcone of D; the proof of Theorem 3.4 implies that $\mathrm{supp}(\Sigma) = D$.

boundary halfspaces, as defined in Appendix A. Then to prove the Lemma it will be enough to show that each of the boundary halfspaces of Σ is contained in the list of halfspaces defining F.

In order to apply Lemma A.1, we first reduce to the case of a full-dimensional, equidimensional simplicial fan in a finite dimensional vector space. For each $m \in \mathbb{Z}_{\geq 0}$, define the subspace $\mathbb{V}_m$ of $\mathbb{V}$ to be

$$\mathbb{V}_m := \{v \in \mathbb{V} \mid v_{i,j} = 0 \text{ unless } -m+i \leq j \leq m+i\}.$$

Note that $\mathbb{V}_m$ contains the Betti diagram of any module with generators in degrees at least $-m$ and with regularity at most m.

Set $\Sigma_m := \Sigma \cap \mathbb{V}_m$ and $F_m := F \cap \mathbb{V}_m$. Observe that $\Sigma_m = \{\mathrm{pos}(C) \mid C \text{ is a chain in } P_m\}$, where $P_m := \{\text{degree sequences } d \mid \pi_d \in \mathbb{V}_m\}$, so by Lemma 2.7, Σ_m is a simplicial fan. Since $\mathbb{V} = \bigcup_{m\geq 0} \mathbb{V}_m$, it is enough to show that $F_m \subseteq \mathrm{supp}(\Sigma_m)$ for all $m \geq 0$.

Next, we define the finite dimensional vector space

$$\overline{\mathbb{V}}_m := \{v \in \mathbb{V}_m \mid v_{i,j} = 0 \text{ unless } i \leq 2\},$$

and consider the projection $\Phi_m \colon \mathbb{V}_m \to \overline{\mathbb{V}}_m$. Since every pure diagram π_d satisfies the functional of type (iiid) in the definition of F, it follows that Φ_m induces an isomorphism of Σ_m onto its image, which we denote by $\overline{\Sigma}_m$. There is also an isomorphism of F_m onto its image $\overline{F}_m$, since the defining halfspaces of F_m contain (iiid). It thus suffices to show that $\overline{F}_m \subseteq \mathrm{supp}(\overline{\Sigma}_m)$ for all m.

We claim that $\overline{\Sigma}_m$ is $(\dim \overline{\mathbb{V}}_m)$-equidimensional. Every maximal chain of degree sequences in P_m begins with $(-m, -m+1, m+n, \dots)$ and ends with $(m, \infty, \infty, \infty, \dots)$. For a fixed maximal chain C, there is a unique $m' \leq m$ such that C is

$$(-m, -m+1, -m+n, \dots) < \cdots < (m', m+1, m'+n, \dots) < (m', \infty, \infty, \dots) < \cdots < (m, \infty, \infty, \dots).$$

From this observation, it follows that

$$|C| = \big((m+m)' + 2(2m+1)\big) + \big(m - m' + 1\big) = 6m + 3.$$

Since the set $\{\pi_d\}$ is linearly independent for $d \in C$ by Lemma 2.7, these diagrams form a basis of $\overline{\mathbb{V}}_m$. It follows that $\overline{\Sigma}_m$ is a $(\dim \overline{\mathbb{V}}_m)$-equidimensional simplicial fan.

We now record a collection of supporting halfspaces which define $\overline{F}_m$:

(i) $\{\epsilon_{i,j} \geq 0\}$ for all $i \geq 0$, $j \in \mathbb{Z} \cap [-m+i, m+i]$;
(ii) $\{\alpha_k \geq 0\}$ for all $k \in \mathbb{Z} \cap [-m, m]$;
(iii) $\{\gamma_{k,m} \geq 0\}$ for all $k \in \mathbb{Z} \cap [-m, m]$, where for $k \in \mathbb{Z} \cap [-m, m]$ we set

$$\gamma_{k,m} := \sum_{j=-m}^{k} \left(2\epsilon_{0,j} - 2\epsilon_{1,j+1} + \epsilon_{2,j+2}\right).$$

To complete the proof, we show that each boundary halfspace of $\overline{\Sigma}_m$ corresponds to a supporting halfspace of $\overline{F}_m$. By Lemma A.1, each boundary halfspace of $\overline{\Sigma}_m$ is determined by (at least one) boundary facet, and hence is determined by some submaximal chain in the poset P_m that is uniquely extended to a maximal chain. The proof of [BS08, Proposition 2.12] applies in our context, showing that each boundary halfspace of $\overline{\Sigma}_m$ depends on only a small part of any submaximal chain to which it corresponds. Namely, such a halfspace is determined by the unique R-degree sequence d that extends a corresponding chain to a maximal one, along with its two neighbors $d' < d''$ in this extended chain, if they exist. We write this data as $\cdots < d' < d^{\frown} < d'' < \cdots$. By direct inspection of P_m (see Figure 1 for the case $m = 1$), the submaximal chains that can be uniquely extended are of the following forms:

(a) $\cdots < d' < d^{\frown} < d'' < \cdots$, where d' and d'' have projective dimension 1 and either $d''_0 - d'_0 = 1$ or $d''_1 - d'_1 = 1$ (but not both). For instance,

$$\cdots < (0, 1, \infty, \infty, \dots) < (0, 2, 3, 4, \dots)^{\frown} < (0, 2, \infty, \infty, \dots) < \cdots.$$

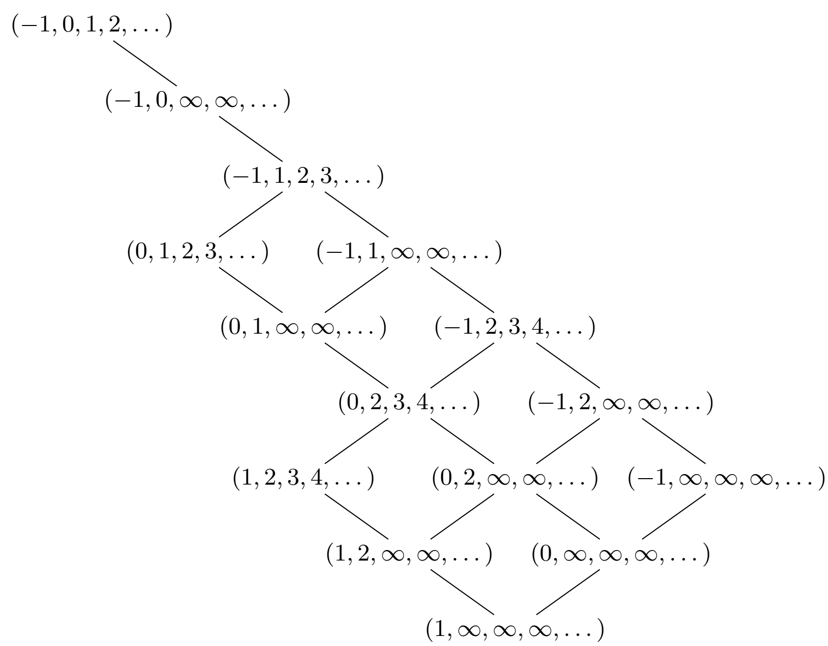

FIGURE 1. The poset of degree sequences whose Betti diagrams lie in $\mathbb{V}_1$.

(b) $\cdots < d' < d^{\wedge} < d'' < \cdots$, where d' and d'' have infinite projective dimension and either $d''_0 - d'_0 = 1$ or $d''_1 - d'_1 = 1$ (but not both). For instance,

$$\cdots < (-1,1,2,3,\dots) < (-1,1,\infty,\infty,\dots)^{\wedge} < (-1,2,3,4,\dots) < \cdots .$$

(c) $\cdots < (d'_0, d'_0+1, \infty, \infty, \dots) < d^{\wedge} < (d'_0+1, d'_0+2, d'_0+3, d'_0+4, \dots) < \cdots$. For instance,

$$\cdots < (0,1,\infty,\infty,\dots) < (0,2,3,4,\dots)^{\wedge} < (1,2,3,4,\dots) < \cdots .$$

(d) $\cdots < d' < d^{\wedge} < d'' < \cdots$, where d' and d'' differ by two in the first entry. For instance,

$$\cdots < (-1,2,3,4,\dots) < (0,2,3,4,\dots)^{\wedge} < (1,2,3,4) < \cdots \text{ or}$$

$$\cdots < (1,\infty,\infty,\infty,\dots) < (2,\infty,\infty,\infty,\dots)^{\wedge} < (3,\infty,\infty,\infty,\dots) < \cdots .$$

(e) $\cdots < (d_0, m+1, m+2, m+3, \dots) < (d_0, m+1, \infty, \infty, \dots)^{\wedge} < (d_0, \infty, \infty, \infty) < \cdots$, for instance

$$\cdots < (0,2,3,4) < (0,2,\infty,\infty)^{\wedge} < (0,\infty,\infty,\infty) < \cdots$$

(f) $\cdots < d' < d^{\wedge}$, where $d' = (m, m+1, \infty, \infty, \dots)$ and $d = (m, \infty, \infty, \infty, \dots)$ is the maximal element of its chain.
(g) $\cdots < d' < d^{\wedge}$, where $d' = (m-1, \infty, \infty, \infty, \dots)$ and $d = (m, \infty, \infty, \infty, \dots)$ is the maximal element of its chain.
(h) $d^{\wedge} < d'' < \cdots$, where d is the minimal element of its chain.

We can show on a case by case basis that each boundary halfspace of $\overline{\Sigma}_m$ (as determined by a submaximal chain from the list above) corresponds to one of the halfspaces defining $\overline{F}_m$; we provide details for a portion of case (a). Consider a submaximal chain C of the form

$$\cdots < (d_0, d_1 - 1, \infty, \dots) < (d_0, d_1, d_2, \dots)^{\wedge} < (d_0, d_1, \infty, \dots) < \cdots ,$$

where $d_2 = d_1 + 1$. Note that $\epsilon^*_{2,d_2}(\pi_c) = 0$ for all $c = (c_0, c_1, \dots) \in C$ because either $c_2 < d_2$ or $c_2 > d_2$. This shows that π_c lies in the hyperplane $\{\epsilon^*_{2,d_2} = 0\}$ for all $c \in C$. Since, in addition, $\epsilon^*_{2,d_2}(\pi_{d_0,d_1,d_2,\dots}) = 1$, it follows that C corresponds to the halfspace $\{\epsilon^*_{0,d_0+1} \geq 0\}$.

Using similar arguments, we see that a submaximal chain of type (a) or (h) corresponds to $\{\epsilon^*_{2,d_2} \geq 0\}$; type (b) corresponds to $\{\alpha_{d_1} \geq 0\}$ and type (e) corresponds to $\{\alpha_{m+1} \geq 0\}$; type (c) or (f) to $\{\gamma_{d'_0,m} \geq 0\}$; and finally, chains of types (d) and (g) correspond to $\{\epsilon^*_{0,m} \geq 0\}$. □

Proof of Theorem 1.2. Let E be the cone spanned by Betti diagrams of extremal modules of finite projective dimension and extremal modules of infinite projective dimension with the stated degree sequences. We see that $D \subseteq E$ by Lemma 2.5, noting that the modules there are extremal of finite projective dimension or of infinite projective dimension with the correct degree sequence. Thus by Theorem 2.4, we have $\mathrm{B}_{\mathbb{Q}}(R) = D$, as desired. □

3. Resolutions over hypersurface rings of embedding dimension 1 and degree at least 2

Set $S := \Bbbk[x]$ and $R := S/\langle x^n \rangle$ for some $n \geq 2$. In this section, we give a full description of the cone of Betti diagrams of R-modules, as well as its implications for the cone of Betti diagrams of maximal Cohen–Macaulay modules over any standard graded hypersurface ring.

Definition 3.1. We say that

$$d = (d_0, d_1, \dots) \in \prod_{i \in \mathbb{N}} (\mathbb{Z} \cup \{\infty\})$$

is an R-degree sequence if it has the form

(i) $d = (d_0, \infty, \infty, \infty, \dots)$ or
(ii) $d = (d_0, d_1, d_2, \dots)$, where $d_0 < d_1$ and $d_{i+2} - d_i = n$ for all $i \geq 0$.

We define a partial order on R-degree sequences as follows: if d has finite projective dimension and d' has infinite dimension, then we say that $d < d'$; otherwise, we use the termwise partial order.

Given an R-degree sequence $d = (d_0, d_1, \dots)$, we define a diagram $\pi_d \in \mathbb{V}$ by

$$(\pi_d)_{i,j} = \begin{cases} 1 & \text{if } j = d_i \neq \infty, \\ 0 & \text{otherwise.} \end{cases}$$

Example 3.2. If $n = 3$, then

$$\pi_{(0,\infty,\infty,\infty,\dots)} = \begin{pmatrix} \vdots & \vdots & \\ - & - & \cdots \\ {}^*1 & - & \cdots \\ - & - & \cdots \\ - & - & \cdots \\ \vdots & \vdots & \end{pmatrix} \quad \text{and} \quad \pi_{(0,1,3,4,\dots)} = \begin{pmatrix} \vdots & \vdots & \vdots & \vdots & \vdots & \\ - & - & - & - & - & \cdots \\ {}^*1 & 1 & - & - & - & \cdots \\ - & - & 1 & 1 & - & \cdots \\ - & - & - & - & 1 & \cdots \\ - & - & - & - & - & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{pmatrix}.$$

To describe the cone $\mathrm{B}_{\mathbb{Q}}(R)$, we define the following functionals on $v \in \mathbb{V}$:

$$\epsilon_{i,j}(v) := v_{i,j}, \qquad \alpha_{i,k}(v) := \epsilon_{i,k}(v) - \epsilon_{i+2,k+n}(v), \qquad \theta_k(v) := \sum_{j \leq k} \epsilon_{2,j}(v) - \sum_{j \leq k-n+1} \epsilon_{1,j}(v),$$

$$\text{and} \qquad \eta_k(v) := \sum_{j \leq k} \left(\epsilon_{1,j}(v) - \epsilon_{2,j+1}(v)\right).$$

Example 3.3. The functional η_3 applied to a Betti diagram $\beta^R(M)$ is given by taking the dot product of $\beta^R(M)$ with the following diagram:

$$\begin{pmatrix} \vdots & \vdots & \vdots & \vdots & & \\ {}^{*}0 & 1 & -1 & 0 & 0 & \cdots \\ 0 & 1 & -1 & 0 & 0 & \cdots \\ 0 & 1 & -1 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & & \end{pmatrix}.$$

Theorem 3.4. *The following cones in $\mathbb{V}$ are equal:*

(i) The cone $\mathrm{B}_{\mathbb{Q}}(R)$ spanned by the Betti diagrams of all finitely generated R-modules.
(ii) The cone D spanned by π_d for all R-degree sequences d.
(iii) The cone F defined as the intersection of the halfspaces
  (a) $\{\epsilon_{i,j} \geq 0\}$ for $i = 0, 1, 2$ and $j \in \mathbb{Z}$;
  (b) $\{\alpha_{0,k} \geq 0\}$ for all $k \in \mathbb{Z}$;
  (c) $\{\theta_k \geq 0\}$ for all $k \in \mathbb{Z}$;
  (d) $\{\eta_k \geq 0\}$ for all $k \in \mathbb{Z}$;
  (e) $\{\pm\alpha_{i,k} \geq 0\}$ for all $i \geq 1, k \in \mathbb{Z}$;
  (f) $\{\pm\eta_\infty \geq 0\}$.

Proof. The equality $\mathrm{B}_{\mathbb{Q}}(R) = D$ follows from the structure theorem of finitely generated modules over principal ideal domains. Using extremal rays, it is also straightforward to check that $\mathrm{B}_{\mathbb{Q}}(R) \subseteq F$. We complete the proof by showing that $F \subseteq D$.

For this final inclusion, note that the proof of Lemma 2.7 also holds in this context, so that $\Sigma = \{\mathrm{pos}(C) \mid C \text{ is a finite chain of } R\text{-degree sequences}\}$ is a simplicial fan; it suffices to prove that $F \subseteq \mathrm{supp}(\Sigma)$. Let $\overline{\mathbb{V}}$ denote the natural projection of $\mathbb{V}$ that sends $v \in \mathbb{V}$ to its first three columns, denoted $v \mapsto \overline{v}$. Denote the respective images of F and Σ under this map by $\overline{F}$ and $\overline{\Sigma}$. For $m \geq 0$, let $P_m := \{\text{degree sequences } d \mid \overline{\pi}_d \in \overline{\mathbb{V}}_m\}$, and define $\overline{\mathbb{V}}_m$, $\overline{\Sigma}_m$, and $\overline{F}_m$ in a manner analogous to the proof of Lemma 2.8. Since every pure diagram π_d satisfies the functionals of types (iiie) and (iiif) in the definition of F, it now suffices to show that $\overline{F}_m \subseteq \mathrm{supp}(\overline{\Sigma}_m)$ for all $m \geq 0$. Note that $\overline{F}_m$ and $\mathrm{supp}(\overline{\Sigma}_m)$ are both contained in the subspace $\overline{\mathbb{W}}_m$ of $\overline{\mathbb{V}}_m$ given by

$$\overline{\mathbb{W}}_m := \left\{ \overline{v} \in \overline{\mathbb{V}}_m \;\middle|\; \eta_\infty(\overline{v}) = 0 \;\text{ and }\; \overline{v}_{2,j} = 0 \text{ for } -m+2 \leq j < n-m. \right\}.$$

We thus view them as objects in there.

Direct computation shows that $\overline{\Sigma}_m \subseteq \overline{\mathbb{W}}_m$ is a full-dimensional, equidimensional simplicial fan. To work towards $\overline{F}_m \subseteq \mathrm{supp}(\overline{\Sigma}_m)$, note that defining halfspaces for $\overline{F}_m \subseteq \overline{\mathbb{W}}_m$ are:

$$\begin{aligned} \{\epsilon_{i,j} \geq 0\} &\quad \text{for } i \in \{0,1,2\} \text{ and } j \in \mathbb{Z} \cap [-m+i, m+i], \\ \{\alpha_{0,j} \geq 0\} &\quad \text{for } j \in \mathbb{Z} \cap [-m, m+2-n], \\ \left\{ \theta_{k,m} := \sum_{j=-m+2}^{k} \epsilon_{2,j} - \sum_{j=-m+1}^{k-n+1} \epsilon_{1,j} \geq 0 \right\} &\quad \text{for } k \in \mathbb{Z} \cap [n-m-1, m+2], \quad \text{and} \\ \left\{ \eta_{km} := \sum_{j=-m+1}^{k} (\epsilon_{1,j} - \epsilon_{2,j+1}) \geq 0 \right\} &\quad \text{for } k \in \mathbb{Z} \cap [-m+1, m+1]. \end{aligned}$$

Each boundary halfspace of $\overline{\Sigma}_m$ depends on certain submaximal chains given by data of the form $\cdots < d' < \widehat{d} < d'' < \cdots$. Such submaximal chains take the following forms:

(a) $\cdots < (d_0, d_1, \dots) < \widehat{(d_0+1, d_1, \dots)} < (d_0+2, d_1, \dots) < \cdots$, where $d_1 < \infty$;
(b) $\cdots < (d_0, d_1, \dots) < \widehat{(d_0, d_1+1, \dots)} < (d_0, d_1+2, \dots) < \cdots$, where $d_1 < \infty$;

(c) $\cdots < (d_0, d_0+1, \ldots) < (d_0, d_0+2, \ldots)\widehat{\ } < (d_0+1, d_0+2, \ldots) < \cdots$;
(d) $\cdots < (d_0, d_0+n-1, \ldots) < (d_0+1, d_0+n-1, \ldots)\widehat{\ } < (d_0+1, d_0+n, \ldots) < \cdots$;
(e) $\cdots < (m-n+2, m, \ldots) < (m-n+2, m+1, \ldots)\widehat{\ } < (-m, \infty, \ldots) < \cdots$;
(f) $\cdots < (m-n+2, m+1, \ldots) < (-m, \infty, \ldots)\widehat{\ } < (-m+1, \infty, \ldots) < \cdots$;
(g) $\cdots < (d_0, \infty, \ldots) < (d_0+1, \infty, \ldots)\widehat{\ } < (d_0+2, \infty, \ldots) < \cdots$;
(h) $(-m, -m+1, \ldots)\widehat{\ } < (-m, -m+2, \ldots) < \cdots$;
(i) $\cdots < (m-1, \infty, \ldots) < (m, \infty, \ldots)\widehat{\ }$.

One may now verify that the boundary halfspaces corresponding to these submaximal chains are, respectively:

(a) $\{\epsilon_{2,d_0+1+n} \geq 0\}$;
(b) $\{\epsilon_{1,d_1+1} \geq 0\}$;
(c) $\{\theta_{d_0+n,m} \geq 0\}$;
(d) $\{\eta_{d_0+n-1_m} \geq 0\}$;
(e) $\{\epsilon_{1,m+1} \geq 0\}$;
(f) $\{\alpha_{0,-m} \geq 0\}$;
(g) $\{\alpha_{0,d_0+1} \geq 0\}$;
(h) $\{\epsilon_{1,-m+1} \geq 0\}$;
(i) $\{\epsilon_{0,m} \geq 0\}$ if $n > 2$ or $\{\alpha_{0,m} \geq 0\}$ if $n = 2$.

As each of these halfspaces appear in our definition of $\overline{F}_m$ above, we obtain $F \subseteq D$, as desired. $\square$

As illustrated by the following corollary, Theorem 3.4 has implications for the study of Betti diagrams of maximal Cohen–Macaulay modules over any standard graded hypersurface ring.

Corollary 3.5. *Let $T = \Bbbk[x_1, \ldots, x_r]/\langle f \rangle$ for any homogeneous f of degree at least 2, and let $\mathrm{B}_{\mathbb{Q}}^{MCM}(T)$ denote the cone of Betti diagrams of maximal Cohen–Macaulay T-modules. Then there is an inclusion*

$$\mathrm{B}_{\mathbb{Q}}^{MCM}(T) \subseteq \mathrm{B}_{\mathbb{Q}}(R),$$

where $R = \Bbbk[x]/\langle x^{\deg(f)} \rangle$. These cones are equal if $\mathrm{char}(\Bbbk)$ does not divide $\deg(f)$.

Proof. Let n be the degree of the homogeneous polynomial f, so that $R = \Bbbk[x]/\langle x^n \rangle$. Recall that $T := \Bbbk[x_1, \ldots, x_r]/\langle f \rangle$, and let M be a maximal Cohen–Macaulay T-module. To show that $\mathrm{B}_{\mathbb{Q}}^{\mathrm{MCM}}(T) \subseteq \mathrm{B}_{\mathbb{Q}}(R)$, we find an R-module M' with the same Betti diagram.

We may assume $\Bbbk$ is infinite by taking a flat extension. Then we find a sequence of M- and R-regular linear forms $(\ell_1, \ldots, \ell_{r-1})$. Note that $T/\langle \ell_1, \ldots, \ell_{r-1} \rangle \cong R$. Applying [Avr98, Corollary 1.2.4], we see that $\beta^T(M) = \beta^{T/\langle \ell_1, \ldots, \ell_{r-1} \rangle}(M/\langle \ell_1, \ldots, \ell_{r-1} \rangle)$, as desired.

For the second statement, assume that $(\deg f, \mathrm{char}(\Bbbk)) = 1$. Since $\mathrm{B}_{\mathbb{Q}}(R) = D$, it is enough to show that for each $\pi_d \in D$, there exists a maximal Cohen–Macaulay T-module M_d such that $\beta^T(M_d) = \pi_d$. If $d = (d_0, \infty, \ldots)$, then $\beta^T(T(-d_0)) = \pi_d$.

Now consider the case that $d = (d_0, d_1, d_0+n, \ldots)$, where without loss of generality $d_0 = 0$ and hence $d_1 < n$. In [BHS88], it is shown that there exists a matrix factorization of f that can be decomposed into a product of n matrices of linear forms. Suppose $A_1 A_2 \cdots A_n$ is such a decomposition. If $M := \mathrm{coker}(A_1 \cdots A_{d_1})$ is presented by this matrix of (d_1)-forms, then it follows that $\beta^T(M) = \pi_d$. Hence $\mathrm{B}_{\mathbb{Q}}^{\mathrm{MCM}}(T) = \mathrm{B}_{\mathbb{Q}}(R)$, as desired. $\square$

4. Multiplicity conjectures and decomposition algorithms

In this section, R denotes a standard graded hypersurface rings of the form $\Bbbk[x]/\langle x^n \rangle$ for any n or $\Bbbk[x,y]/\langle q \rangle$, where q is any homogeneous quadric. We first note that Theorem 1.3 follows from the proofs of Lemma 2.8 and Theorem 3.4, as they provide the desired simplicial structure.

This simplicial structure gives rise to a greedy decomposition algorithm of Betti diagrams into pure diagrams, as in [ES09, §1]. The key fact is that, since the cone $\mathrm{B}_{\mathbb{Q}}(R)$ is simplicial, for any

module M, there is a finite chain of degree sequences $d_1 < \ldots < d_n$ such that $\beta^R(M)$ is a positive linear combination of the π_{d_i}. And as noted in Lemma 2.7, the diagram π_{d_i} has a nonzero entry in a position in which, for all $j > i$, π_{d_j} has a zero entry. We now present a detailed example to illustrate the algorithm.

Example 4.1. Let $R = S/\langle x^2\rangle$ and $M = \operatorname{coker}\begin{pmatrix} x & xy^2 & y^4 \\ 0 & y^3 & xy^3 \end{pmatrix}$. Then we have

$$\beta^R(M) = \begin{pmatrix} 2 & 1 & 1 & 1 & \cdots \\ - & - & - & - & \cdots \\ - & 1 & - & - & \cdots \\ - & 1 & 1 & 1 & \cdots \end{pmatrix}.$$

We decompose $\beta^R(M)$ by first considering the minimal R-degree sequence that could possibly contribute to $\beta^R(M)$, which is $(0, 1, 2, 3, \ldots)$. We then subtract $\frac{1}{2}\pi_{(0,1,2,3,\ldots)}$, as this is the largest multiple that can be removed while remaining inside $\mathrm{B}_{\mathbb{Q}}(R)$. This yields

$$\beta^R(M) - \frac{1}{2}\pi_{(0,1,2,3,\ldots)} = \begin{pmatrix} \frac{3}{2} & - & - & - & \cdots \\ - & - & - & - & \cdots \\ - & 1 & - & - & \cdots \\ - & 1 & 1 & 1 & \cdots \end{pmatrix}.$$

We next subtract one copy of $\pi_{(0,3,\infty,\infty,\ldots)}$, to obtain

$$\beta^R(M) - \frac{1}{2}\pi_{(0,1,2,3,\ldots)} - \pi_{(0,3,\infty,\infty,\ldots)} = \begin{pmatrix} \frac{1}{2} & - & - & - & \cdots \\ - & - & - & - & \cdots \\ - & - & - & - & \cdots \\ - & 1 & 1 & 1 & \cdots \end{pmatrix}.$$

Note that the remaining Betti diagram equals $\frac{1}{2}\pi_{(0,4,5,6,\ldots)}$. In particular, $\beta^R(M)$ lies in the face corresponding to the chain

$$(0, 1, 2, 3, \ldots) < (0, 3, \infty, \infty, \ldots) < (0, 4, 5, 6, \ldots).$$

The existence of these simplicial structures also gives rise to R-analogues of the Herzog–Huneke–Srinivasan Multiplicity Conjectures. We say that an R-degree sequence d is compatible with a Betti diagram $\beta^R(M)$ if $\beta^R_{i,d_i}(M) \neq 0$ when $d_i < \infty$.

Corollary 4.2. *Let M be an R-module generated in a single degree. Let $\underline{d} = (\underline{d_0}, \underline{d_1}, \ldots)$ be the minimal R-degree sequence compatible with $\beta^R(M)$, and let $\overline{d} = (\overline{d_0}, \overline{d_1}, \ldots)$ be the maximal R-degree sequence compatible with $\beta^R(M)$.*

(i) We have

$$e(M) \leq \beta^R_0(M) \cdot e(\pi_{\overline{d}}).$$

(ii) If $\overline{d_1} < \infty$, then

$$\beta^R_0(M) \cdot e(\pi_{\underline{d}}) \leq e(M) \leq \beta^R_0(M) \cdot e(\pi_{\overline{d}}),$$

with equality on either side if and only if $\underline{d} = \overline{d}$.

Proof. Since M is generated in a single degree, we may assume that $\underline{d_0} = 0$. By Theorem 1.3, there is a unique chain $\underline{d} = d^0 < d^1 < \cdots < d^s = \overline{d}$ for which

$$\beta^R(M) = \sum_{i=0}^{s} a_i \pi_{d^i}. \tag{4.1}$$

If $\overline{d} = (0, \infty, \infty, \ldots)$, then M has dimension 1 and $e(M) = a_s e(\pi_{\overline{d}})$. Since $a_s \leq \beta^R_{0,0}(M)$, this proves (i) in the case that $\overline{d_1} = \infty$.

We now assume that $\overline{d_1} = \infty$, and prove (ii), which implies (i) for this remaining case. We first compute the multiplicity of π_d for any R-degree sequence d of the form $d = (0, d_1, d_2, d_3, \dots)$ with $d_1 < \infty$. We consider separately the cases $\infty = d_2 = d_3 = \cdots$ and $d_i = d_1 + i - 1$ for all $i \geq 2$.

We may assume that $\Bbbk$ is infinite by taking a flat extension. For the first case, we may assume after a possible change of coordinates that y is a nonzero divisor on R. Then the Betti diagram of $R/\langle y^{d_1}\rangle$ equals $\pi_{(0,d_1,\infty,\infty,\dots)}$, and hence

$$e(\pi_{(0,d_1,\infty,\infty,\dots)}) = e(R/\langle y^{d_1}\rangle) = 2d_1.$$

For the remaining case, the Betti diagram of $R/\langle y^{d_1}, xy^{d_1-1}\rangle$ equals $\pi_{(0,d_1,d_1+1,d_1+2,\dots)}$, and hence

$$e(\pi_{(0,d_1,d_1+1,d_1+2,\dots)}) = e(R/\langle y^{d_1}, xy^{d_1-1}\rangle) = 2d_1 - 1.$$

Note that, since $\overline{d_1} < \infty$, every pure diagram π_{d^i} arising in the decomposition (4.1) satisfies $d^i_0 = 0$ and $d^i_1 < \infty$. Therefore

$$e(\pi_{d^0}) < e(\pi_{d^1}) < \cdots < e(\pi_{d^s}).$$

By convexity, this implies (ii). □

Appendix A. Convex Geometry

In this appendix, we provide background on some convex geometry. The curious reader may turn to [Zie95, Chapters 1,2,7] for even further details.

Let V be a $\mathbb{Q}$-vector space. A subset $C \subseteq V$ is a convex cone if it closed under addition and multiplication by elements of $\mathbb{Q}_{\geq 0}$. For a subset $B \subseteq V$, $\operatorname{pos}(B)$ denotes the positive hull of B, defined as $\operatorname{pos}(B) := \left\{\sum_{b\in B} a_b b \mid a_b \in \mathbb{Q}_{\geq 0}\right\}$, which is clearly a cone. A ray is the $\mathbb{Q}_{\geq 0}$-span on an element of V. A ray in a positive hull $\operatorname{pos}(B)$ is an extremal ray of $\operatorname{pos}(B)$ if it does not lie in $\operatorname{pos}(B \setminus \{b\})$.

We say C is a n-dimensional simplicial cone if $C = \operatorname{pos}(B)$ for a set of n linearly independent vectors B. An m-dimensional face of C is a subset of the form $\operatorname{pos}(B')$, for B' a subset of m vectors of B. A facet of C is an $(n-1)$-dimensional face.

A simplicial fan Σ is a collection of simplicial cones $\{C_i\}$ such that $C_i \cap C_j$ is a face of both C_i and C_j for all i, j. We refer to $\bigcup_i C_i \subseteq V$ as the support of Σ. We say that a subset Σ of V has the structure of a simplicial fan if Σ is the support of some simplicial fan.

A simplicial fan Σ that is a finite union of cones is m-equidimensional if each maximal cone has dimension m. A facet of an equidimensional fan is a facet of any maximal cone, and it is a boundary facet if it is contained in exactly one maximal cone. If $\dim V$ is finite and Σ is $(\dim V)$-equidimensional, then each boundary facet F determines a unique, up to scalar, functional $L\colon V \to \mathbb{Q}$ such that L vanishes along F and is nonnegative on the (unique) maximal cone containing F; we refer to the halfspace $\{L \geq 0\}$ as a boundary halfspace of the fan.

Simplicial fan structures that come from posets arise throughout this paper. Let P be a poset and assume that there is a map $\Phi\colon P \to V$ such that $\Phi(p_1), \dots, \Phi(p_s)$ is linearly independent in V for all chains $p_1 < \ldots < p_s$ in P and such that the union of simplicial cones

$$\Sigma(P, \Phi) := \{\operatorname{pos}(\{\Phi(p_1), \dots, \Phi(p_s)\}) \mid s \in \mathbb{Z}_{\geq 0} \text{ and } p_1 < \cdots < p_s \text{ is a chain in } P\}$$

is a simplicial fan. (When P is finite, this fan is referred to as a geometric realization of P. In our cases, P is the poset of R-degree sequences, and Φ is the map $d \mapsto \pi_d \in \mathbb{V}$.) If $\dim V$ is finite, then maximal cones of $\Sigma(P, \Phi)$ are in bijection with maximal chains in P, and submaximal chains in P are in bijection with facets of $\Sigma(P, \Phi)$.

Lemma A.1. *Let V be an m-dimensional $\mathbb{Q}$-vector space, P be a finite poset, $\Phi\colon P \to V$ as above, and $\Sigma(P,\Phi)$ be an m-equidimensional simplicial fan. Then there is a bijective map:*

$$\left\{\begin{matrix}\text{submaximal chains of } P \text{ that} \\ \text{lie in a unique maximal chain of } P\end{matrix}\right\} \longrightarrow \left\{\begin{matrix}\text{boundary facets} \\ \text{of } \Sigma(P,\Phi)\end{matrix}\right\}$$

that is given by sending the submaximal chain $p_1 < \cdots < p_{m-1}$ to $\operatorname{pos}(\{\Phi(p_1),\dots,\Phi(p_m)\})$.

In addition, since $p_1 < \cdots < p_{m-1}$ lies in a unique maximal cone, there is a unique $q \in P$ which extends this to a maximal chain. The boundary halfspace determined by this submaximal chain is the halfspace $\{L \geq 0\}$, where $L(\Phi(p_i)) = 0$ and $L(\Phi(q)) > 0$. Though more than one submaximal chain may determine the same boundary halfspace, each boundary halfspace corresponds to at least one such chain. □

Example A.2. Let P be the poset from Figure 1. We continue with the notation of the proof of Lemma 2.8, letting D'_1 be the simplicial fan on P. Since P has 12 maximal chains, it follows that D'_1 is the union of 12 simplicial cones (of dimension 9). Consider the maximal chain corresponding to the lower left boundary of Figure 1: there are 7 submaximal chains that uniquely extend to this maximal chain. More precisely, there are respectively $0, 2, 2, 0, 1, 1, 0, 1$ such submaximal chains of type (a)–(h).

Although simplicial fans are not necessarily convex, we can always construct a convex cone from a simplicial fan.

Lemma A.3. *Let V be an m-dimensional $\mathbb{Q}$-vector space, and let Σ be an m-equidimensional simplicial fan. Let $\{\{L_k \geq 0\}\}$ be the set of boundary halfspaces of Σ. The convex cone $\bigcap_k \{L_k \geq 0\}$ is a subset of the support of Σ.*

Proof. The arguments in the proof of Theorem 2.15 of [Zie95] show that $\bigcap_k \{L_k \geq 0\}$ is the largest convex cone contained in the support of Σ. □

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